Malaysian Journal of Mathematical Sciences 19(1): 269–287 (2025)
https://doi.org/10.47836/mjms.19.1.14

Malaysian Journal of Mathematical Sciences

Journal homepage: https://mjms.upm.edu.my

On Graph Entropy Measures Based on the Number of Dominating and Power Dominating Sets

Geethu Kuriachan [1] and Parthiban, A.* [1]

[1] *Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore, Tamil Nadu, India*

E-mail: parthiban.a@vit.ac.in
**Corresponding author*

Received: 14 June 2024
Accepted: 22 November 2024

Abstract

This article examines graph entropy measures that depend on the number of dominating and power-dominating sets. To quantify the structural complexity of a graph structure, one uses graph entropies. It is easy to compute these properties for smaller networks, and if reliable approximations are developed, similar metrics can also be used for larger graphs. Using various graph invariants, many graph entropy measures have already been established and computed. So, in this work, a new graph entropy measure, namely, power domination entropy, using the power domination polynomial, is introduced. The domination and power domination polynomials of graphs are used to determine the number of dominating and power dominating sets. Let $D(G,\xi)$ represent the collection of all dominating sets of G with size ξ, $d_\xi(G)=|D(G,\xi)|$, and γ_s be the total number of dominating sets of G. Then, the domination entropy of G with n nodes is defined as $I_{dom}(G)=-\sum\limits_{\xi=1}^{n}\frac{d_\xi(G)}{\gamma_s(G)}\,\log\left(\frac{d_\xi(G)}{\gamma_s(G)}\right)$. The domination and power domination entropies for a few graphs are further computed. Following that, a comparison between the domination and power domination entropies of several graphs is provided.

Keywords: graph entropy measures; domination; domination polynomial; power domination; power domination polynomial.

1 Introduction

The characterization of the complexity of network systems is one of the major difficulties faced by modern research [15, 26]. There are two main methods for measuring complexity: computable and uncomputable. Traditional computable complexity measurements are graph entropies, which are derived from Shannon's entropy formula [28]. The concept of graph entropy, introduced by Rashevsky [25] in 1955, is based on partitioning vertices into equivalence classes according to their degrees. Mowshowitz [22] later expanded this framework by incorporating graph automorphism groups. Building on these foundations, Mowshowitz [20], in 1968, applied information theory to explore both chemical and mathematical structures, establishing a broader relevance of entropy measures in these fields. The automorphism group of a graph's symmetry serves as the foundation for this measure. Larger entropy values result from poor symmetry and high element variety in a complex system [7, 5], and vice versa. Using graph entropies, one can compare the complicated graph structures. As noted in [29], it could be useful to define new graph entropies based on different graph invariants.

Domination entropy, an entropy measure derived from the dominating sets, is presented in [27]. This motivated us to derive the domination entropy of more graph structures. In this paper, the domination entropy is computed for certain graphs. Sometimes two different graph structures give the same entropy measure, which is one of the most challenging problems in this area. So, finding a suitable graph entropy measure to analyze the structure of a graph is more important. Since the power domination problem is computationally demanding, in this paper, a new graph entropy measure, namely, power domination entropy, using the power dominating sets, is introduced and studied for various graphs, besides the comparison between domination entropy and power domination entropy of some graphs. And we conclude that domination entropy has more discriminating power than power domination entropy. In order to define this entropy, Dehmer's information functional approach [12] is used. Throughout this paper, we refer to the dominating set as DS, the domination polynomial as DP, and the power domination polynomial as PDP for the sake of convenience.

1.1 Basic definitions

Let G be a simple graph with the node set $V(G)$ and the line set $E(G)$. The path, cycle, complete, wheel and star graph of order n are denoted by P_n, C_n, K_n, W_n and S_n, respectively. The n-barbell graph Bar_n on $2n$ nodes is formed by joining two copies of a complete graph K_n by a single line, shown in Figure 1.

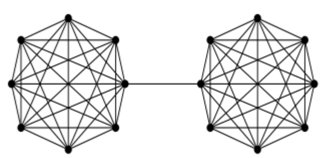

Figure 1: The barbell graph of order 16, Bar_8.

The graph with nodes $a_1, a_2, \ldots, a_{2s}$ is known as the cocktail party graph $CP(s)$ of order $2s$ if all pairs of distinct nodes form lines in this graph, except for the pairs $\{a_1, a_2\}, \{a_3, a_4\}, \ldots, \{a_{2s-1}, a_{2s}\}$. The r-book graph B_r is constructed by bonding r copies of C_4 along a common line, see Figure 2.

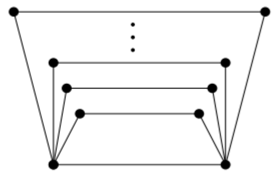

Figure 2: The book graph B_r.

The graph created by joining n copies of C_3 with a common node is called the friendship (or Dutch-Windmill) graph F_n. A cone graph, also known as a generalized wheel graph, $C_{m,n}$, is defined by the graph join $C_m + \overline{K_n}$. A fan graph $F_{m,n}$ is $\overline{K_m} + P_n$. The graph formed by taking m copies of K_n with a node in common is called the (m, n)-windmill graph W_n^m. This means that $mK_{n-1} + K_1$ is isomorphic to the (m, n)-windmill graph.

Two nodes $x, y \in V(G)$ are adjacent, or neighbors, if $\{x, y\} \in E(G)$.

Definition 1.1. [30] *If every node outside of $D \subseteq V(G)$ is neighbor to at least one node inside D, then D is a DS of G. $\gamma(G)$ represents the domination number of G, which is the lowest of the cardinalities of the DS of G.*

Refer [1, 16] for further information on domination in graphs.

Definition 1.2. [30] *If a set Q of nodes observes every node in G by the following rules:*

- *all nodes in Q and all neighbors of nodes in Q are observed,*
- *whenever a node v in G is observed and all but one of its neighbors, say w, are observed, then the node w is also observed by v,*

then it is considered a power dominating set (PDS). The lowest size of a PDS of G is its power domination number, $\gamma_p(G)$.

Definition 1.3. [2] *The collection of all dominating sets of G with size ξ is represented by $D(G, \xi)$ and let $d_\xi(G) = |D(G, \xi)|$. Then the DP, $D(G, y)$ of G is,*

$$D(G, y) = \sum_{\xi=\gamma(G)}^{|V(G)|} d_\xi(G) y^\xi.$$

Definition 1.4. [2] *Let $\gamma_s(G)$ represents the total number of sets that dominate G. Then,*

$$\gamma_s(G) = \sum_{\xi=\gamma(G)}^{|V(G)|} d_\xi(G).$$

Or, $\gamma_s(G)$ is obtained by putting y=1 in the domination polynomial.

Definition 1.5. [9] *The collection of all PDS of G with size ξ is represented by $P(G, \xi)$, and $p_\xi(G) = |P(G, \xi)|$. Thus, the following equation introduces the PDP, $P(G, y)$ of G:*

$$P(G, y) = \sum_{\xi=\gamma_p(G)}^{|V(G)|} p_\xi(G) y^\xi.$$

Denote the total number of PDS by $(\gamma_p)_s$. Consider the cycle $C_4 : x_1x_2x_3x_4$. To power dominate C_4, one node that is enough to power dominate the other three nodes. Thus $\gamma_p(C_4) = 1$. The set of PDS of C_4 with cardinality one is $P(G, 1) = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}\}$, and $p_1(G) = 4$. Furthermore, the set of PDS of C_4 with cardinality two is $P(G, 2) = \{\{x_1, x_2\}, \{x_1, x_3\}, \{x_1, x_4\}, \{x_2, x_3\}, \{x_2, x_4\}, \{x_3, x_4\}\}$, and $p_2(G) = 6$, the set of PDS of C_4 with cardinality three is $P(G, 3) = \{\{x_1, x_2, x_3\}, \{x_1, x_2, x_4\}, \{x_1, x_3, x_4\}, \{x_2, x_3, x_4\}\}$, and $p_3(G) = 4$.

Finally, the PDS of C_4 with cardinality four is $P(G, 4) = \{x_1, x_2, x_3, x_4\}$, and $p_4(G) = 1$. Consequently, $P(C_4, y) = y^4 + 4y^3 + 6y^2 + 4y$. It is noted that there are fifteen PDS in total for C_4, i.e, $(\gamma_p)_s(C_4) = 15$, and it is derived straight from the power domination polynomial's coefficients. As a result, the following definition can be stated:

Definition 1.6. [9] *For a given G, suppose $(\gamma_p)_s$ represents the total number of PDS. Then,*

$$(\gamma_p)_s = \sum_{\xi=\gamma_p(G)}^{|V(G)|} p_\xi(G).$$

Or, $(\gamma_p)_s$ is obtained by putting y=1 in the power domination polynomial.

Definition 1.7. [12] *Dehmer's information functional approach is utilised to define the entropy of G. "Given $T = \{t_1, t_2, \ldots, t_r\}$, allow $f : T \to R_+$ to be an information functional, such that T is a set of elements of G. Then, the following definition of entropy is given:*

$$\begin{aligned} I_f(G) &= -\sum_{i=1}^{r} \frac{f(t_i)}{\sum\limits_{j=1}^{r} f(t_j)} \log \left(\frac{f(t_i)}{\sum\limits_{j=1}^{r} f(t_j)} \right) \\ &= \log\left(\sum_{i=1}^{r} f(t_i)\right) - \frac{\sum\limits_{i=1}^{r} f(t_i) \log f(t_i)}{\sum\limits_{j=1}^{r} f(t_j)}, \end{aligned}$$

where logarithmic phrases have base 2".

Definition 1.8. [27] *The domination entropy of G with n nodes is defined by,*

$$I_{dom}(G) = I_f(G) = -\sum_{\xi=1}^{n} \frac{d_\xi(G)}{\gamma_s(G)} \log\left(\frac{d_\xi(G)}{\gamma_s(G)}\right).$$

In this case, $d_\xi(G) = 0$ for every $\xi < \gamma(G)$, and $d_{n-1}(G) = n$. With a new information functional, the power domination entropy can now be defined.

Example 1.1. *If P is a Petersen graph, then $I_{dom}(P) = 2.05754$.*

Solution: The DP of the petersen graph P is,

$$\begin{aligned} D(P, x) =& x^{10} + \binom{10}{9}x^9 + \binom{10}{8}x^8 + \binom{10}{7}x^7 + \left(\binom{10}{6} - 10\right)x^6 + \\ & \left(\binom{10}{5} - 60\right)x^5 + 75x^4 + 10x^3. \end{aligned}$$

Thus, by Definition 1.4,

$$\gamma_s(P) = \sum_{i=3}^{10} d_i(P)$$
$$= 1 + \binom{10}{9} + \binom{10}{8} + \binom{10}{7} + \binom{10}{6} - 10 + \binom{10}{5} - 60 + 75 + 10$$
$$= 443.$$

Then,

$$I_{dom}(P) = \log\ 443 - \frac{1}{443}\left(\sum_{i=3}^{10} d_i(G)\ \log\left(d_i(G)\right)\right)$$
$$= 2.05754.$$

Definition 1.9. *By taking the information functional $f = p_\xi(G)$ in the Definition 1.7 , the power domination entropy is defined as,*

$$I_{pdom}(G) = I_f(G) = -\sum_{\xi=1}^{n} \frac{p_\xi(G)}{(\gamma_p)_s(G)}\ \log\left(\frac{p_\xi(G)}{(\gamma_p)_s(G)}\right),$$

where $p_\xi(G) = 0$ *for all* $\xi < \gamma_p(G)$, $p_{n-1}(G) = n$, *&* $p_n(G) = 1$.

2 Related Works

In the past few years, numerous investigations have been conducted to ascertain the complexity of the networks. Graph entropy measurements have been widely applied in transdisciplinary research, encompassing fields such as biology, chemistry, and information science [8]. Bonchev [4] provided foundational insights into using information-theoretic indices for characterizing chemical structures, establishing their importance in quantitative analysis. Extending these concepts, Bonchev and Buck [6] explored quantitative measures of network complexity, highlighting their relevance in diverse scientific domains. More recently, Chen et al. [11] investigated network aesthetics based on symmetry, demonstrating the evolving applications of entropy measures in understanding complex networks. Mowshowitz [21] applied information theory to analyze chemical and mathematical structures, pioneering the use of entropy measures in graph theory.

Randic and Plavsic [24] further characterized molecular complexity, emphasizing the significance of structural properties in chemistry. Their work introduced novel methods to quantify molecular complexity and demonstrated its relevance to quantum chemical applications [23]. Building on these ideas, Mowshowitz and Dehmer [20] revisited graph entropy and complexity, offering a contemporary perspective on its role in understanding graph structures. Most graph entropies are derived from the Shannon entropy to determine the graphs' complexity. Numerous graph entropy measures are available in the literature, and they are derived from the graph's order, degree sequence, distance, characteristic polynomials, and other graph polynomials [14]. Recently, there has been introduction of graph entropies, which are associated with molecular descriptors. The definition of graph entropy measures [10] and their further investigation [29] are based on matchings and independent sets. Mowshowitz and Dehmer [17] also look into various relationships between the graph's complexity and Hosoya entropy. The study [13] applies various graph entropies to quantify structural complexity across different application areas, including biological, social, and technological networks.

3 Main Results

This section includes the domination and power domination entropy of some graphs.

3.1 Domination entropy of some graphs

Alikhani and Peng [2] established the concept of domination polynomials. Domination polynomials were also found for $K_n \square K_2$, B_n, Bar_n and $CP(n)$ [18, 19]. This study finds the domination entropy of these graphs using the domination polynomials.

Theorem 3.1. *For a book graph B_s with cardinality $2s+2$, the domination entropy,*

$$I_{dom}(B_s) = \log(3^{s+1} + 2^{2s} - 2) - \frac{1}{3^{s+1} + 2^{2s} - 2}\Bigg[\Bigg(\sum_{\xi=2}^{s-1}\binom{2s}{\xi-2}\log\binom{2s}{\xi-2}\Bigg) +$$
$$\left(s2^s + \binom{2s}{s-2} - 2\right)\log\left(s2^s + \binom{2s}{s-2} - 2\right) + \sum_{\xi=s+1}^{2s+1}\left(\binom{s}{2s+1-\xi}2^{(2s+2)-\xi}\right.$$
$$\left. + \binom{s}{2s-\xi}2^{\xi} + \binom{2s}{\xi-2}\right)\log\left(\binom{s}{2s+1-\xi}2^{(2s+2)-\xi} + \binom{s}{2s-\xi}2^{\xi} + \binom{2s}{\xi-2}\right)\Bigg].$$

Proof. From [19],

$$D(B_s, y) = (y^2 + 2y)^s(2y+1) + y^2(y+1)^{2s} - 2y^s, \tag{1}$$

and $\gamma(B_s) = 2$. By Definition 1.4, $\gamma_s(B_s) = \sum_{\xi=2}^{2s+2} d_\xi(G) = 3^{s+1} + 2^{2s} - 2$. To find the coefficient of y^ξ in (1), we first expand $(y^2+2y)^s$ using the binomial theorem as follows,

$$(y^2+2y)^s = \sum_{j=0}^{s}\binom{s}{j}(y^2)^{s-j}(2y)^j = \sum_{j=0}^{s}\binom{s}{j}2^j y^{2s-j}.$$

Then,

$$(y^2+2y)^s(2y+1) = \sum_{j=0}^{s}\binom{s}{j}2^j(2y^{2s-j+1} + y^{2s-j}) = \sum_{j=0}^{s}\binom{s}{j}2^{j+1}y^{2s-j+1} + \sum_{j=0}^{s}\binom{s}{j}2^j y^{2s-j}.$$

Now, consider $y^2(y+1)^{2s}$. By using the binomial theorem to expand $(y+1)^{2s}$, we have,

$$y^2(y+1)^{2s} = \sum_{j=0}^{2s}\binom{2s}{j}y^{j+2}.$$

To compute the coefficient of y^ξ, examine each of these expanded terms. First, for $\sum_{j=0}^{s}\binom{s}{j}2^{j+1}y^{2s-j+1}$, substitute $2s-j+1=\xi$, which gives $j = 2s+1-\xi$. The coefficient of y^ξ in this term is $\binom{s}{2s+1-\xi}2^{2s+2-\xi}$, valid for $s+1 \le \xi \le 2s+1$.

Next, in $\sum_{j=0}^{s}\binom{s}{j}2^j y^{2s-j}$, substitute $2s-j=\xi$, giving $j=2s-\xi$. The coefficient of y^{ξ} in this case is $\binom{s}{2s-\xi}2^{\xi}$, valid for $s\le\xi\le 2s$.

Finally, in $\sum_{j=0}^{2s}\binom{2s}{j}y^{j+2}$, substitute $j+2=\xi$, yielding $j=\xi-2$. The coefficient of y^{ξ} is $\binom{2s}{\xi-2}$, valid for $2\le\xi\le 2s+1$.

From the above discussions, one can get,

$$
\begin{aligned}
d_{\xi}(B_s) &= \binom{2s}{\xi-2}, \quad \text{for } 2\le\xi\le s-1,\\
d_s(B_s) &= s2^s+\binom{2s}{s-2}-2,\\
d_{\xi}(B_s) &= \binom{s}{2s+1-\xi)}2^{(2s+2)-\xi}+\binom{s}{2s-\xi}2^{\xi}+\binom{2s}{\xi-2}, \quad \text{for } s+1\le\xi\le 2s+1.
\end{aligned}
$$

It is known that $d_{2s+2}(B_s)=1$. Thus,

$$
\begin{aligned}
I_{dom}(B_s) = &\log(3^{s+1}+2^{2s}-2)-\frac{1}{3^{s+1}+2^{2s}-2}\Bigg[\Bigg(\sum_{\xi=2}^{s-1}\binom{2s}{\xi-2}\,\log\,\binom{2s}{\xi-2}\Bigg)+\\
&\left(s2^s+\binom{2s}{s-2}-2\right)\log\,\left(s2^s+\binom{2s}{s-2}-2\right)+\sum_{\xi=s+1}^{2s+1}\Bigg(\binom{s}{2s+1-\xi)}2^{(2s+2)-\xi}\\
&+\binom{s}{2s-\xi}2^{\xi}+\binom{2s}{\xi-2}\Bigg)\log\,\Bigg(\binom{s}{2s+1-\xi}2^{(2s+2)-\xi}+\binom{s}{2s-\xi}2^{\xi}+\binom{2s}{\xi-2}\Bigg)\Bigg].
\end{aligned}
$$

□

Theorem 3.2. *For a Bar_m with order $2m$, the domination entropy,*

$$
\begin{aligned}
I_{dom}(Bar_m) = &\log((2^m-1)^2)-\frac{1}{(2^m-1)^2}\times\\
&\sum_{j=2}^{2m}\left(\left[\binom{2m}{2m-j}-2\binom{m}{m-j}\right]\log\left[\binom{2m}{2m-j}-2\binom{m}{m-j}\right]\right).
\end{aligned}
$$

Proof. From [19], $D(Bar_m,y)=((1+y)^m-1)^2$ and $\gamma(Bar_m)=2$. By Definition 1.4, $\gamma_s(Bar_m)=\sum_{j=2}^{2m}d_j(Bar_m)=(2^m-1)^2$. To find the coefficient of y^j from $D(Bar_m,y)$, we start by expanding $((1+y)^m-1)^2$ as follows,

$$((1+y)^m-1)^2=\left(\sum_{k=1}^{m}\binom{m}{k}y^k\right)^2=\sum_{k=1}^{m}\sum_{l=1}^{m}\binom{m}{k}\binom{m}{l}y^{k+l}.$$

Therefore, to get the coefficient of y^j, sum the products of binomial coefficients where $k+l=j$ within the valid range for k and l. This is given by,

$$\sum_{k=1}^{m}\binom{m}{k}\binom{m}{j-k}.$$

Using the convolution identity for binomial coefficients, we know that,

$$\sum_{k=0}^{m}\binom{m}{k}\binom{m}{j-k}=\binom{2m}{j}.$$

However, since k and $j-k$ must both be at least 1, we need to exclude the cases where $k=0$ or $j-k=0$, which results in exclusion of $2\binom{m}{j}$.

Thus, the coefficient of y^j is $\binom{2m}{j}-2\binom{m}{j}$, for $2\le j\le 2m$. Using the symmetry property, $\binom{2m}{j}=\binom{2m}{2m-j}$ and $\binom{m}{j}=\binom{m}{m-j}$, where $\binom{m}{m-j}=0$ for $j>m$.

From the above, one can get $d_j(Bar_m)=\binom{2m}{2m-j}-2\binom{m}{m-j}$for $2\le j\le 2m$, where $\binom{m}{m-j}=0$ for $j>m$.

Thus, the domination entropy of Bar_m is,

$$I_{dom}(Bar_m)=\log((2^m-1)^2)-\frac{1}{(2^m-1)^2}\times \sum_{j=2}^{2m}\left(\left[\binom{2m}{2m-j}-2\binom{m}{m-j}\right]\log\left[\binom{2m}{2m-j}-2\binom{m}{m-j}\right]\right).$$

□

Theorem 3.3. *For a $K_r\square K_2$, the domination entropy,*

$$I_{dom}(K_r\square K_2)=\log((2^r-1)^2+2)-\frac{1}{(2^r-1)^2+2}\left[\sum_{\substack{j=1\\ j\neq r}}^{2r}\left(\binom{2r}{2r-j}-2\binom{r}{r-j}\right)\log\left(\binom{2r}{2r-j}-2\binom{r}{r-j}\right)+\left(\binom{2r}{r}\log\binom{2r}{r}\right)\right].$$

Proof. It is known that $D(K_r\square K_2,y)=((1+y)^r-1)^2+2y^r$. From the Definition 1.4, $\gamma_s(K_r\square K_2)=\sum_{j=1}^{2r}d_j(G)=(2^r-1)^2+2$. To determine the coefficients of the polynomial $D(K_r\square K_2,y)$, we first expand $((1+y)^r-1)^2$ as follows,

$$((1+y)^r-1)^2=\left(\sum_{k=1}^{r}\binom{r}{k}y^k\right)^2=\sum_{m=2}^{2r}\left(\sum_{k=1}^{m-1}\binom{r}{k}\binom{r}{m-k}\right)y^m.$$

Then,

$$D(K_r \square K_2, y) = \sum_{m=2}^{2r} \left(\sum_{k=1}^{m-1} \binom{r}{k} \binom{r}{m-k} \right) y^m + 2y^r.$$

Therefore, the coefficient of y^j for $j \neq r$ is $\sum_{k=1}^{j-1} \binom{r}{k}\binom{r}{j-k} = \binom{2r}{j} - \binom{r}{j}$ and for $j = r$, the coefficient is $\sum_{k=1}^{r-1} \binom{r}{k}\binom{r}{r-k} + 2 = \binom{2r}{r} - \binom{r}{r} + 2 = \binom{2r}{r} + 1.$

From the above, one can determine the number of dominating sets as,

$$d_j(K_r \square K_2) = \begin{cases} \binom{2r}{2r-j} - \binom{r}{r-j}, & \text{for } j \neq r, \\ \binom{2r}{r} + 1, & \text{for } j = r, \end{cases}$$

where for $j > r$, $\binom{r}{r-j} = 0$. Therefore, the domination entropy of $K_r \square K_2$ is,

$$\begin{aligned} I_{dom}(K_r \square K_2) &= \log(\gamma_s(K_r \square K_2)) - \frac{1}{\gamma_s(K_r \square K_2)} \sum_{j=1}^{2r} d_j(K_r \square K_2)\ \log(d_j(K_r \square K_2)) \\ &= \log((2^r - 1)^2 + 2) - \frac{1}{(2^r-1)^2 + 2} \Bigg[\sum_{\substack{j=1 \\ j \neq r}}^{2r} \left(\binom{2r}{2r-j} - 2\binom{r}{r-j} \right) \\ &\quad \log\left(\binom{2r}{2r-j} - 2\binom{r}{r-j} \right) + \left(\binom{2r}{r} \log \binom{2r}{r} \right) \Bigg]. \end{aligned}$$

□

Theorem 3.4. *Let $CP(k)$ be a cocktail party graph with order $2k$. Then,*

$$I_{dom}(CP(k)) = \log\,(2^{2k} - 2k - 1) - \frac{1}{2^{2k} - 2k - 1} \left(\sum_{i=2}^{2k} \binom{2k}{i}\ \log\ \binom{2k}{i} \right).$$

Proof. The domination polynomial of $CP(k)$ is $D(CP(k), y) = (1+y)^{2k} - 2ky - 1$. Therefore, $\gamma_s(CP(k)) = 2^{2k} - 2k - 1$. By expanding $(1+y)^{2k}$ using the binomial theorem, we get $(1+y)^{2k} - 1 = \sum_{i=1}^{2k} \binom{2k}{i} y^i$. Then, $(1+y)^{2k} - 2ky - 1 = \sum_{i=2}^{2k} \binom{2k}{i} y^i$. Thus, the coefficient of y^i in $D(CP(k), y)$ is $\binom{2k}{i}$.

From the above, $d_1(CP(k)) = 0$ and $d_i(CP(k)) = \binom{2k}{i}$ for $2 \leq i \leq 2k$. Thus,

$$\begin{aligned} I_{dom}(CP(k)) &= \log\,(2^{2k} - 2k - 1) - \frac{1}{2^{2k} - 2k - 1} \left(\sum_{i=2}^{2k} d_i(G)\ \log\ d_i(G) \right) \\ &= \log\,(2^{2k} - 2k - 1) - \frac{1}{2^{2k} - 2k - 1} \left(\sum_{i=2}^{2k} \binom{2k}{i}\ \log\ \binom{2k}{i} \right). \end{aligned}$$

□

3.2 Power domination entropy of some graphs

The power domination polynomial of K_n, P_n, C_n, W_n, S_n and $\overline{K_n}$ are obtained in [12] and in this section, power domination entropy of these graphs are obtained. Also, power domination polynomial and power domination entropy of some other families of graphs such as W_n^m, F_n, DP_n, $C_{m,n}$, and $F_{m,n}$ are derived.

Theorem 3.5. [12] $P(K_n, y) = P(P_n, y) = P(C_n, y) = P(W_n, y) = (y+1)^n - 1$.

Theorem 3.6. *For a K_n,*

$$I_{pdom}(K_n) = \log\,(2^n - 1) - \frac{1}{2^n - 1}\sum_{i=1}^{n}\binom{n}{i}\ \log\ \binom{n}{i}.$$

Proof. $P(K_n, y) = (1+y)^n - 1$ is the PDP of K_n. It follows that $(\gamma_p)_s = 2^n - 1$. The coefficient of y^i in the polynomial $P(K_n, y)$ can be determined by examining its expansion $P(K_n, y) = \sum_{i=1}^{n}\binom{n}{i} y^i$. Here, for $1 \le i \le n$, the coefficient of y^i is $\binom{n}{i}$.

Thus, for $1 \le i \le n$, $p_i(G) = \binom{n}{i}$. Furthermore, $\gamma_p(K_n) = 1$.

Hence,

$$\begin{aligned} I_{pdom}(K_n) &= \log((\gamma_p)_s) - \frac{1}{(\gamma_p)_s}\sum_{i=1}^{n} p_i(K_n)\ \log(p_i(K_n)) \\ &= \log\,(2^n - 1) - \frac{1}{2^n - 1}\sum_{i=1}^{n}\binom{n}{i}\ \log\ \binom{n}{i}. \end{aligned}$$

□

Remark 3.1. *Similarly, one can derive the power domination entropy of P_n, C_n, and W_n as derived in above theorem since the power domination polynomials of K_n, P_n, C_n, and W_n are the same.*

Theorem 3.7. *For a $\overline{K_n}$, $I_{pdom}(\overline{K_n}) = -n\ \log\ n$.*

Proof. Since $\overline{K_n}$ is the complement of K_n, $\overline{K_n}$ be the empty graph on n nodes. Then, $P(\overline{K_n}, y) = y^n$. Since the sum of the coefficients of $P(\overline{K_n}, y)$ is 1, $(\gamma_p)_s(\overline{K_n}) = 1$ and $\gamma_p(\overline{K_n}) = 1$. Thus,

$$I_{pdom}(\overline{K_n}) = -n\ \log\ n.$$

□

Theorem 3.8. *For a S_r,*

$$\begin{aligned} I_{pdom}(S_r) =& \log(2^{r-1} + r) - \frac{1}{2^{r-1} + r}\Bigg[\sum_{i=2}^{r-3}\binom{r-1}{i-1}\log\binom{r-1}{i-1} + \\ & \Big(2(r-1)\Big)\ \log\Big(2(r-1)\Big) + 2\ \log\ 2\Bigg]. \end{aligned}$$

Proof. The power domination polynomial of S_r is $P(S_r, y) = y(y+1)^{r-1} + y^{r-1} + (r-1)y^{r-2}$ for $r \geq 3$. Therefore, $(\gamma_p)_s(S_r) = 2^{r-1} + r$. To expand $P(S_r, y)$, we begin by applying the binomial expansion to $y(y+1)^{r-1}$, yielding $\sum_{i=1}^{r} \binom{r-1}{i-1} y^i$. Then,

$$P(S_r, y) = \sum_{i=1}^{r} \binom{r-1}{i-1} y^i + y^{r-1} + (r-1)y^{r-2},$$

which implies that,

$$P(S_r, y) = y + \sum_{i=2}^{r-3} \binom{r-1}{i-1} y^i + 2(r-1)y^{r-2} + 2y^{r-1} + y^r.$$

Thus from the above, $p_1(S_r) = 1, p_i(S_r) = \binom{r-1}{i-1}$ for $2 \leq i \leq r-3$, $p_{r-2}(S_r) = 2(r-1)$, and $p_{r-1}(S_r) = 2$. Hence,

$$I_{pdom}(S_r) = \log(2^{r-1} + r) - \frac{1}{2^{r-1} + r}\left[\sum_{i=2}^{r-3} \binom{r-1}{i-1} \log \binom{r-1}{i-1} + \Big(2(r-1)\Big) \log \Big(2(r-1)\Big) + 2 \log 2\right].$$

□

Theorem 3.9. [2] *Let n_1 and n_2 be the nodes of graphs G_1 and G_2, respectively. If $n_j > 1$, then for every $j \in \{1, 2\}$, I_j equals the number of isolates of G_j; otherwise, it equals zero. Assume $G = G_1 \vee G_2$. After that, $P(G; y) = (1 + I_1/y)P(G_1; y) + (1 + I_2/y)P(G_2; y) + ((y+1)^{n_1} - 1)((y+1)^{n_2} - 1)$.*

Theorem 3.10. $P(W_n^m, y) = ((y+1)^{n-1} - 1)^m + y(y+1)^{mn-m}$.

Proof. Since $W_n^m \simeq mK_{n-1} \vee K_1$, by Theorem 3.9 it follows that,

$$\begin{aligned}
P(W_n^m, y) &= P(mK_{n-1} \vee K_1; y) \\
&= (1+0)P(mK_{n-1}, y) + (1+0)P(K_1, y) + ((y+1)^{mn-m} - 1)((y+1)^1 - 1) \\
&= ((y+1)^{n-1} - 1)^m + y(y+1)^{mn-m}.
\end{aligned}$$

□

Theorem 3.11. *For a F_n,*

$$P(F_n, y) = (y^2 + 2y)^n + y(y+1)^{2n},$$

and

$$\begin{aligned}
I_{pdom}(F_n) = &\log(3^n + 2^{2n}) - \frac{1}{3^n + 2^{2n}}\left[\sum_{i=1}^{n} \binom{2n}{i-1} \log \binom{2n}{i-1} + \right. \\
&\left. \sum_{i=n+1}^{2n} \left(\binom{n}{i-n} 2^{2n-i} + \binom{2n}{i-1}\right) \log \left(\binom{n}{i-n} 2^{2n-i} + \binom{2n}{i-1}\right)\right].
\end{aligned}$$

Proof. According to Theorem 3.9, since $F_n \simeq K_1 \vee nK_2$,

$$\begin{aligned}
P(F_n, y) &= (1+0)P(K_1, y) + (1+0)P(nK_2, y) + ((y+1)^1 - 1)((y+1)^{2n} - 1) \\
&= (y^2 + 2y)^n + y(y+1)^{2n}.
\end{aligned}$$

Since, $P(F_n, y) = (y^2+2y)^n + y(y+1)^{2n}$, $(\gamma_p)_s(F_n) = 3^n + 2^{2n}$. First, we expand $(y^2+2y)^n$ using the binomial theorem, which gives $(y^2+2y)^n = \sum_{k=0}^{n} \binom{n}{k} y^{2k} \cdot 2^{n-k} \cdot y^{n-k} = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} y^{n+k}$. Thus, the powers of y from this expansion range from y^n (when $k=0$) to y^{2n} (when $k=n$), and the coefficient of y^{n+k} is $\binom{n}{k} 2^{n-k}$.

Next, we expand $y(y+1)^{2n}$. Then, $y(y+1)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} y^{k+1}$. Therefore, the powers of y from this expansion range from y^1 (when $k=0$) to y^{2n+1} (when $k=2n$), and the coefficient of y^{k+1} is $\binom{2n}{k}$. For $1 \le i \le n$, so the coefficient of y^i is $\binom{2n}{i-1}$. For $n+1 \le i \le 2n$, both expansions contribute and the coefficient of y^i is $\binom{n}{i-n} 2^{2n-i} + \binom{2n}{i-1}$.

Thus, $p_i(F_n)$ are given by,

$$p_i(F_n) = \begin{cases} \binom{2n}{i-1}, & \text{for } 1 \le i \le n, \\ \binom{n}{i-n} 2^{2n-i} + \binom{2n}{i-1}, & \text{for } n+1 \le i \le 2n. \end{cases}$$

Then,

$$I_{pdom}(F_n) = \log(3^n + 2^{2n}) - \frac{1}{3^n + 2^{2n}} \Bigg[\sum_{i=1}^{n} \binom{2n}{i-1} \log \binom{2n}{i-1} + \sum_{i=n+1}^{2n} \left(\binom{n}{i-n} 2^{2n-i} + \binom{2n}{i-1} \right) \log \left(\binom{n}{i-n} 2^{2n-i} + \binom{2n}{i-1} \right) \Bigg].$$

□

Theorem 3.12. *For $C_{r,s}$,*

$$P(C_{r,s}, y) = ((y+1)^r - 1)((y+1)^s) + (y+s)y^{s-1},$$

and

$$I_{pdom}(C_{r,s}) = \log((2^r-1)2^s + (s+1)) - \frac{1}{(2^r-1)2^s + (s+1)} \Bigg[\sum_{\substack{i=1 \\ i \ne s-1, s}}^{rs} \left(\binom{r+s}{i} \right) \log \left(\binom{r+s}{i} \right) + \left(\binom{r+s}{s-1} + s \right) \log \left(\binom{r+s}{s-1} + s \right) + \left(\binom{r+s}{s} + 1 \right) \log \left(\binom{r+s}{s} + 1 \right) \Bigg].$$

Proof. Since $C_{r,s} \simeq C_r \vee \overline{K_s}$, by Theorem 3.9 it follows that,

$$\begin{aligned} P(C_{r,s}, y) &= P(C_r \vee \overline{K_s}; y) \\ &= (1+0)P(C_r, y) + (1 + \frac{s}{y}) P(\overline{K_s}, y) + ((y+1)^r - 1)((y+1)^s - 1) \\ &= ((y+1)^r - 1)(y+1)^s + (y+r)y^{s-1}. \end{aligned}$$

Since, $P(C_{r,s}, y) = ((y+1)^r - 1)((y+1)^s) + (y+s)y^{s-1}$, $(\gamma_p)_s(C_{r,s}) = (2^r - 1)2^s + (s+1)$. To find the coeffient of y^i in $P(C_{r,s}, y)$, first calculating expansion of $((y+1)^r - 1)((y+1)^s)$ using the binomial theorem. Then,

$$((y+1)^r - 1)((y+1)^s) = \left(\sum_{i=1}^{r} \binom{r}{i} y^i\right)\left(\sum_{j=0}^{s} \binom{s}{j} y^j\right) = \sum_{i=1}^{r}\sum_{j=0}^{s} \binom{r}{i}\binom{s}{j} y^{i+j}.$$

Additionally, expanding $(y+s)y^{s-1}$ adds terms y^s and sy^{s-1} to the final expression.

The coefficients of y^i are derived as follows: for $1 \le i \le rs$ and $i \ne s, s-1$, the coefficient is $\sum_{j=0}^{i} \binom{r}{i-j}\binom{s}{j} = \binom{r+s}{i}$. Now, the coefficient of y^{s-1} is $\sum_{j=0}^{i} \binom{r}{s-1-j}\binom{s}{j} + s = \binom{r+s}{s-1} + s$.

Finally, the coefficient of y^s is $\sum_{j=0}^{i} \binom{r}{s-j}\binom{s}{j} + 1 = \binom{r+s}{s} + 1$. Thus, through the use of the PDP expansion, $p_i(C_{r,s}) = \binom{r+s}{i}$ for $1 \le i \le rs$ & $i \ne s, s-1$, $P_{s-1}(C_{r,s}) = \binom{r+s}{s-1} + s$ and $P_s(C_{r,s}) = \binom{r+s}{s} + 1$.

Hence,

$$\begin{aligned} I_{pdom}(C_{r,s}) =& \log((\gamma_p)_s) - \frac{1}{(\gamma_p)_s}\sum_{i=1}^{rs} p_i(C_{r,s}) \ \log(p_i(C_{r,s})) \\ =& \log((2^r-1)2^s + (s+1)) - \frac{1}{(2^r-1)2^s + (s+1)}\Bigg[\sum_{\substack{i=1 \\ i\ne s-1,s}}^{rs} \left(\binom{r+s}{i}\right) \log\left(\binom{r+s}{i}\right) \\ &+ \left(\binom{r+s}{s-1} + s\right) \log\left(\binom{r+s}{s-1} + s\right) + \left(\binom{r+s}{s} + 1\right) \ \log\ \left(\binom{r+s}{s} + 1\right)\Bigg]. \end{aligned}$$

□

Theorem 3.13. *For* $F_{m,n}$*,*

$$P(F_{m,n}, y) = (y+m)y^{m-1} + ((y+1)^n - 1)(y+1)^m,$$

and

$$\begin{aligned} I_{pdom}(F_{m,n}) =& \log\ ((2^n-1)2^m + m + 1) - \frac{1}{(2^n-1)2^m + m + 1}\Bigg[\sum_{\substack{i=1 \\ i\ne m-1,m}}^{mn} \left(\binom{n+m-1}{m-1} + m - 1\right) \\ & \log\ \left(\binom{n+m-1}{m-1} + m - 1\right) + \left(\binom{mn}{n-1} + m - 1\right) \log\left(\binom{n+m}{m}\right) + \binom{n+m}{m} \\ & \log\ \binom{n+m}{m}\Bigg]. \end{aligned}$$

Proof. Since $F_{m,n} \simeq \overline{K_m} \vee P_n$, by Theorem 3.9 it follows that,

$$\begin{aligned} P(F_{m,n}, y) &= P(\overline{K_m} \vee P_n; y) \\ &= (1 + \frac{m}{y})P(\overline{K_m}, y) + (1+0)P(P_n, y) + ((y+1)^m - 1)((y+1)^n - 1) \\ &= (y+m)y^{m-1} + ((y+1)^n - 1)(y+1)^m. \end{aligned}$$

Since $P(F_{m,n}, y) = (y+m)y^{m-1} + ((y+1)^n - 1)(y+1)^m$, $(\gamma_p)_s(F_{m,n}) = (2^n - 1)2^m + m + 1$. Now, expand $P(F_{m,n}, y)$ to get the coefficients of y^i. First, expand $(y+1)^n - 1 = ny + \binom{n}{2}y^2 + \cdots + y^n$ and $(y+1)^m = 1 + my + \binom{m}{2}y^2 + \cdots + y^m$. Therefore, $P(F_{m,n}, y) = y^m + my^{m-1} + (ny + \binom{n}{2}y^2 + \cdots + y^n)(1 + my + \binom{m}{2}y^2 + \cdots + y^m)$.

Then the coefficients of y^i in $P(F_{m,n}, y)$ are derived as follows:

for $i \neq m-1, m$, the coefficient of y^i is $\sum_{k=0}^{i} \binom{n}{k}\binom{m}{i-k} - 1 = \binom{n+m}{i} - 1$.

For $i = m-1$, the coefficient of y^i is $m + \sum_{k=0}^{m-1} \binom{n}{k}\binom{m}{(m-1)-k} - 1 = m + \binom{n+m-1}{m-1} - 1$.

For $i = m$, the coefficient of y^i is $1 + \sum_{k=0}^{m} \binom{n}{k}\binom{m}{m-k} - 1 = \sum_{k=0}^{m} \binom{n}{k}\binom{m}{k} = \binom{n+m}{m}$.

Therefore from the above, we get,

$$\begin{aligned} p_i(F_{m,n}) &= \binom{n+m}{i} - 1, \text{for } 1 \leq i \leq mn, i \neq m-1, m, \\ p_i(F_{m,n}) &= \binom{n+m-1}{m-1} + m - 1, \text{for } i = m-1, \\ p_i(F_{m,n}) &= \binom{n+m}{m}, \text{for } i = m. \end{aligned}$$

Also $\gamma_p(F_{m,n}) = 1$. Hence,

$$\begin{aligned} I_{pdom}(F_{m,n}) =& \log((\gamma_p)_s) - \frac{1}{(\gamma_p)_s} \sum_{i=1}^{mn} p_i(F_{m,n}) \ \log(p_i(F_{m,n})) \\ =& \log\left((2^n - 1)2^m + m + 1\right) - \frac{1}{(2^n-1)2^m + m + 1}\Bigg[\sum_{\substack{i=1 \\ i \neq m-1, m}}^{mn} \left(\binom{n+m-1}{m-1} + m - 1\right) \\ & \log\left(\binom{n+m-1}{m-1} + m - 1\right) + \left(\binom{mn}{n-1} + m - 1\right) \log\left(\binom{n+m}{m}\right) + \binom{n+m}{m} \\ & \log\binom{n+m}{m}\Bigg]. \end{aligned}$$

□

4 Comparison of Domination and Power Domination Entropies

Figure 3 displays the graphs used in the numerical experiments in an ascending order based on their topological complexity (TC), which is determined by adding up all of the adjacencies of each subgraph [3]. It seems to be a more reliable and is a non-entropy based method of assessing complexity that makes use of the subgraph count concept. I_{dom} values are extracted from [27]. In addition, Table 1 lists the power domination entropy alongside the domination entropy for each of the 21 graphs. The power dominating polynomials of the graphs shown in Figure 3 can be utilised.

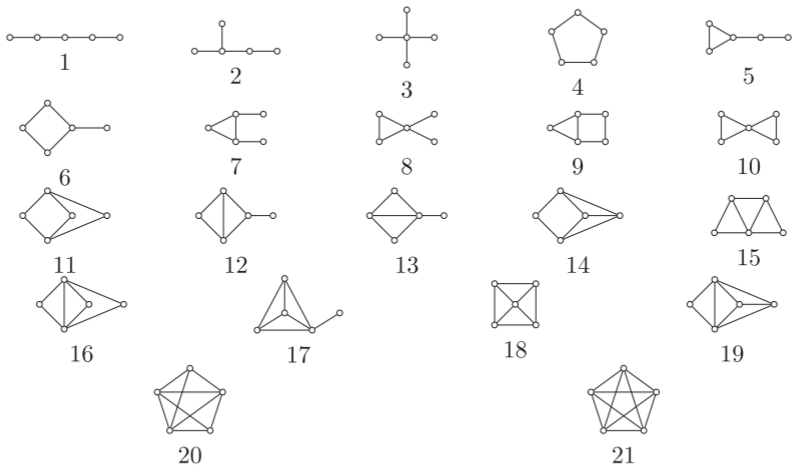

Figure 3: 21 simple connected graphs on 5 nodes is arranged in order of topological complexity (TC).

$$
\begin{aligned}
P(1,y) &= y^5 + 5y^4 + 10y^3 + 10y^2 + 5y,\\
P(2,y) &= y^5 + 5y^4 + 10y^3 + 9y^2 + y,\\
P(3,y) &= y^5 + 5y^4 + 10y^3 + 4y^2 + y,\\
P(4,y) &= y^5 + 5y^4 + 10y^3 + 10y^2 + 5y,\\
P(5,y) &= y^5 + 5y^4 + 10y^3 + 9y^2 + 3y,\\
P(6,y) &= y^5 + 5y^4 + 10y^3 + 10y^2 + 4y,\\
P(7,y) &= y^5 + 5y^4 + 10y^3 + 10y^2 + 3y,\\
P(8,y) &= y^5 + 5y^4 + 10y^3 + 8y^2 + y,\\
P(9,y) &= y^5 + 5y^4 + 10y^3 + 10y^2 + 5y,\\
P(10,y) &= y^5 + 5y^4 + 10y^3 + 8y^2 + y,\\
P(11,y) &= y^5 + 5y^4 + 10y^3 + 10y^2 + 2y,\\
P(12,y) &= y^5 + 5y^4 + 10y^3 + 10y^2 + 4y,\\
P(13,y) &= y^5 + 5y^4 + 10y^3 + 10y^2 + 4y,\\
P(14,y) &= y^5 + 5y^4 + 10y^3 + 10y^2 + 4y,\\
P(15,y) &= y^5 + 5y^4 + 10y^3 + 10y^2 + 5y,\\
P(16,y) &= y^5 + 5y^4 + 10y^3 + 10y^2 + 2y,\\
P(17,y) &= y^5 + 5y^4 + 10y^3 + 10y^2 + 5y,
\end{aligned}
$$

$$P(18, y) = y^5 + 5y^4 + 10y^3 + 10y^2 + 5y,$$
$$P(19, y) = y^5 + 5y^4 + 10y^3 + 10y^2 + 4y,$$
$$P(20, y) = y^5 + 5y^4 + 10y^3 + 10y^2 + 5y,$$
$$P(21, y) = y^5 + 5y^4 + 10y^3 + 10y^2 + 5y.$$

The power domination entropy is calculated using the formula defined in Definition 1.9.

Example 4.1. *Let G_{10} be the tenth graph in Figure 3. Then, $(\gamma_p)_s(G_{10}) = 25$ and,*

$$I_{pdom}(G_{10}) = -\frac{1}{25}\log\left(\frac{1}{25}\right) - \frac{8}{25}\log\left(\frac{8}{25}\right) - \frac{10}{25}\log\left(\frac{10}{25}\right) - \frac{5}{25}\log\left(\frac{5}{25}\right) - \frac{1}{25}\log\left(\frac{1}{25}\right)$$
$$= 1.89.$$

Table 1: I_{dom}, I_{pdom} and TC of 21 graphs.

Graphs	I_{dom}	I_{pdom}	TC
1	1.712	2.062	60
2	1.688	1.879	76
3	2.022	1.877	100
4	1.704	2.062	160
5	1.741	2.0176	172
6	1.713	2.0386	290
7	1.712	2.0027	212
8	1.952	1.8907	230
9	1.719	2.062	482
10	1.890	1.8907	292
11	1.719	1.9485	504
12	1.741	2.0386	511
13	1.927	2.0386	566
14	1.709	2.0386	1278
15	1.890	2.062	1316
16	1.719	1.9485	1394
17	1.923	2.062	1396
18	1.863	2.062	3216
19	1.709	2.0386	3290
20	2.001	2.062	7806
21	2.060	2.062	18180

Table 1 offers numerous outcomes that can be obtained. Since the domination polynomial and power domination polynomial of G_{10} are equal, all terms of I_{dom} and I_{pdom} are equal.

Therefore, one can get that $I_{dom}(G_{10}) = I_{pdom}(G_{10})$. The value of I_{pdom} presented in the table shows that third graph in Figure 3 attains the minimum entropy because the number of power dominating sets with cardinality 1 and 2 is very low compared to other graphs. The graphs 1, 4, 9, 15, 17, 18, 20, 21, 5, 6, 7, 12, 13, 14, and 19 in Figure 3 have the largest power dominating entropy measures in the increasing order. The power domination number of these graphs is one. It is evident that the highest value of the power domination entropy measure is present when the number of power dominating sets with varying cardinalities is nearly equal.

In the case of a complete graph, I_{dom} and I_{pdom} are very high and the same, since the domination polynomial and power domination polynomial of a complete graph are same. The graphs 1 and 7 have the same I_{dom} but different I_{pdom}. This suggests that there is some similarity in the complexity of the two graphs. Graphs with low complexity are invariably associated with low entropy values.

Examining Table 1 makes it clear that all of the measures aside from TC are very degenerate and inadequate at differentiating between structures. The goal of mathematical chemists is to find a suitable measure that can distinguish each graph individually. This implies that the values of non-isomorphic networks should differ. If two non-isomorphic graphs are of the same complexity, it can be difficult to find a suitable complexity measure that would yield a result that is precisely the same for both non-isomorphic graphs.

5 Conclusions

The domination entropy of cocktail party, barbell, and book graphs is determined in this work. The power domination entropy is defined using the power dominating sets of graphs and derived for some graphs, such as cycle, star, path, complete, and so on. The applications of the domination and power domination polynomials, respectively, yield the number of dominating and power dominating sets in a graph

There are certain unresolved issues for further research. Investigating the domination and power domination entropy, similar graph structures can be found in the future, and their domination and power domination polynomials can be derived. More recently, the roots of domination and power domination polynomials have been studied and can also be studied for other graphs. Furthermore, the domination and power domination entropies of maximal and minimal graphs can also be studied.

Acknowledgement The authors would like to thank the anonymous reviewers and editor for their detailed comments that improved the presentation and quality of the paper.

Conflicts of Interest The authors declare no conflict of interest.

References

[1] H. A. Ahangar & M. Khaibari (2017). Graphs with large Roman domination number. *Malaysian Journal of Mathematical Sciences*, *11*(1), 71–81.

[2] S. Alikhani & Y. h. Peng (2014). Introduction to domination polynomial of a graph. *Ars Combinatoria*, *114*, 257–266. https://doi.org/10.48550/arXiv.0905.2251.

[3] D. G. Bonchev (1995). Kolmogorov's information, Shannon's entropy, and topological complexity ofmolecules. *Bulgarian chemical communications*, *28*(3-4), 567–582.

[4] D. G. Bonchev (1983). *Information Theoretic Indices for Characterization of Chemical Structures*. Research Studies Press, Hertfordshire.

[5] D. G. Bonchev (2009). *Encyclopedia of Complexity and Systems Science*, volume 5, chapter Information Theoretic Complexity Measures., pp. 4820–4838. Springer, New York. https://doi.org/10.1007/978-0-387-30440-3_285.

[6] D. G. Bonchev & G. A. Buck (2005). *Complexity in Chemistry, Biology, and Ecology*, chapter Quantitative Measures of Network Complexity, pp. 191–235. Springer, Boston, MA. https://doi.org/10.1007/0-387-25871-X_5.

[7] D. G. Bonchev & D. Rouvray (2003). *Complexity in Chemistry: Introduction and Fundamentals* volume 7 of *Mathematical Chemistry*. Taylor and Francis, Boca Raton, Florida.

[8] D. G. Bonchev & D. Rouvray (2005). *Complexity in Chemistry, Biology, and Ecology*. Mathematical and Computational Chemistry. Springer, New York. https://doi.org/10.1007/b136300.

[9] B. Brimkov, R. Patel, V. Suriyanarayana & A. Teich. Power domination polynomials of graphs. arXiv: Combinatorics 2018. https://doi.org/10.48550/arXiv.1805.10984.

[10] S. Cao, M. Dehmer & Z. Kang (2017). Network entropies based on independent sets and matchings. *Applied Mathematics and Computation*, *307*, 265–270. https://doi.org/10.1016/j.amc.2017.02.021.

[11] Z. Chen, M. Dehmer, F. Emmert-Streib, A. Mowshowitz & Y. Shi (2017). Toward measuring network aesthetics based on symmetry. *Axioms*, *6*(2), Article ID: 12. https://doi.org/10.3390/axioms6020012.

[12] M. Dehmer (2008). Information processing in complex networks: Graph entropy and information functionals. *Applied Mathematics and Computation*, *201*(1-2), 82–94. https://doi.org/10.1016/j.amc.2007.12.010.

[13] M. Dehmer, F. Emmert Streib, Z. Chen, X. Li & Y. Shi (2016). *Mathematical Foundations and Applications of Graph Entropy* volume 6 of *Quantitative and Network Biology*. John Wiley & Sons, Weinheim. https://doi.org/10.1002/9783527693245.

[14] M. Dehmer & A. Mowshowitz (2011). A history of graph entropy measures. *Information Sciences*, *181*(1), 57–78. https://doi.org/10.1016/j.ins.2010.08.041.

[15] J. L. Green, A. Hastings, P. Arzberger, F. J. Ayala, K. L. Cottingham, K. Cuddington, F. Davis, J. A. Dunne, M.-J. Fortin, L. Gerber et al. (2005). Complexity in ecology and conservation: mathematical, statistical, and computational challenges. *BioScience*, *55*(6), 501–510. https://doi.org/10.1641/0006-3568(2005)055[0501:CIEACM]2.0.CO;2.

[16] R. Hasni, D. A. Mojdeh & S. A. Bakar (2024). Domination (totally) dot-critical of Harary graphs. *Malaysian Journal of Mathematical Sciences*, *18*(1), 1–7. https://doi.org/10.47836/mjms.18.1.01.

[17] H. Hosoya (1971). Topological index. a newly proposed quantity characterizing the topological nature of structural isomers of saturated hydrocarbons. *Bulletin of the Chemical Society of Japan*, *44*(9), 2332–2339. https://doi.org/10.1246/bcsj.44.2332.

[18] S. Jahari & S. Alikhani (2016). On $\mathcal{D}$-equivalence classes of some graphs. *Bulletin of the Georgian National Academy of Sciences*, *10*(1), 12–19. https://doi.org/10.48550/arXiv.1511.00159.

[19] T. Kotek, J. Preen & P. Tittmann. Domination polynomials of graph products. arXiv: Combinatorics 2013. https://doi.org/10.48550/arXiv.1305.1475.

[20] A. Mowshowitz (1968). Entropy and the complexity of graphs: I. An index of the relative complexity of a graph. *The Bulletin of Mathematical Biophysics*, *30*, 175–204. https://doi.org/10.1007/BF02476948.

[21] A. Mowshowitz & M. Dehmer (2012). Entropy and the complexity of graphs revisited. *Entropy*, *14*(3), 559–570. https://doi.org/10.3390/e14030559.

[22] A. Mowshowitz & V. Mitsou (2009). *Entropy, Orbits, and Spectra of Graphs*, chapter 1, pp. 1–22. John Wiley & Sons, Ltd, Weinheim. https://doi.org/10.1002/9783527627981.ch1.

[23] M. Randić & D. Plavšić (2002). On the concept of molecular complexity. *Croatica Chemica Acta*, *75*(1), 107–116.

[24] M. Randić & D. Plavsić (2003). Characterization of molecular complexity. *International Journal of Quantum Chemistry*, *91*(1), 20–31. https://doi.org/10.1002/qua.10343.

[25] N. Rashevsky (1955). Life, information theory, and topology. *The Bulletin of Mathematical Biophysics*, *17*, 229–235. https://doi.org/10.1007/BF02477860.

[26] R. W. Rycroft & D. E. Kash (1999). *The Complexity Challenge: Technological Innovation for the 21st Century*. Science, Technology, and the International Political Economy. Pinter, London.

[27] B. Şahin (2022). New network entropy: The domination entropy of graphs. *Information Processing Letters*, *174*, Article ID: 106195. https://doi.org/10.1016/j.ipl.2021.106195.

[28] C. E. Shannon & W. Weaver (1964). *The Mathematical Theory of Communication*. University of Illinois Press, Urbana.

[29] P. Wan, X. Zhang, B. Wu & X. Li (2020). On graph entropy measures based on the number of independent sets and matchings. *Information Sciences*, *516*, 491–504. https://doi.org/10.1016/j.ins.2019.11.020.

[30] M. Zhao, L. Kang & G. J. Chang (2006). Power domination in graphs. *Discrete Mathematics*, *306*(15), 1812–1816. https://doi.org/10.1007/978-3-030-51117-3_16.